ŠESTEROKUT
SOLUTIONS

2021

SADRŽAJ

Part I

Fizika

DIMENZIONALNA ANALIZA I "SLOBODAN" PAD

At times, to reach a valid physical law, one can use a tool called dimensional analysis. An example of the use of this tool is presented on free fall.

When a sphere of mass m and radius r is dropped into free fall from height h, it requires a certain amount of time to reach the ground. The total falling time can depend on the values of the parameters describing the body and the forces present in the system. Supposing there is no air drag and no buoyancy, the only parameter describing the forces on the sphere is the gravitational acceleration g. Thus, we can try to write the falling time as:

$$t = A m^{\alpha} h^{\beta} g^{\gamma} r^{\delta} \,,$$

where $\alpha, ..., \delta$ are unknown coefficients and A is some constant.

- Determine the possible combinations of values of $\alpha, ..., \delta$.
  S. (2 points) Since the left side of the equation has the dimension of time, thr right side exponents are restricted:

  $$[s] = [1][kg]^{\alpha}[m]^{\beta}[\frac{m}{s^2}]^{\gamma}[m]^{\delta}$$

  immediately giving $\alpha = 0$ and $\gamma = -1/2$ and hence $\beta + \delta = 1/2$.

- Perform a virtual experiment using a simulation tool available at `shorturl.at/mKRY4`. Select "lab", then select "cannonball" and turn off air drag. The launch angle should be set to 90°. Leave the gravitational acceleration as it is. Prove that $\alpha = 0$ by performing measurements for spheres of varying masses, but of same radius.
  S. (3 points) Any setup that allows the student to measure the falling time is fine. One example is to select the parameters in the problem and use the measuring tools to determine the times for spheres of various masses. In this case, measurements show that, given all the other conditions remain equal, the falling time does not depend on the mass of the sphere. This proves that there is no dependence $t(m)$ and hence α=0.

- Prove that $\delta = 0$ by performing measurements for spheres of varying radii, but of same masses. How are β and γ now related?
  S. (3 points) Same as the previous paragraph. Measurements show that there is no dependence $t(r)$ and hence δ=0. In that case $\beta = 1/2$ since $\gamma = -1/2$ and $\beta + \delta$ is equal to 1/2.

- Determine the constant A by performing additional measurements and write down the complete expression for the time the body requires to fall to the ground when dropped from height h.
  S. (2 points) Performing additional measurements (these measurements have no error and thus one measurement is enough) give $A = \sqrt{2}$, giving the full expression $t = \sqrt{2h/g}$.

Now we will include air drag in our model. We can assume that the total drag depends on the sphere radius r, the current sphere velocity v and its mass m. A simple model for air drag is given by

$$F = -kv^{\alpha},$$

where α is some unknown coefficient and k is a coefficient that depends on the properties of the sphere.

- Perform a virtual experiment using a simulation tool available at `shorturl.at/mKRY4`. Select "vectors", and include the velocity and force vectors in the simulation. The launch angle should be set to 90°. Determine if $\alpha = 1$ or $\alpha = 2$ by performing measurements.
  S. (3 points) The students can make screenshots (or press pause) of the projectile motion and measure the lengths of the velocity and force vectors. The author made the following set of measures of the pairs of velocity and drag force (in pixels): (168,21),(231,42),(331,88) and (408,134). An exponent fit can be made on these measurements but since we only have to determine if $\alpha = 1$ or 2, this can be simplified. Scaling all the pairs with the first one we obtain (1,1), (1.4,2), (2,4.2) and (2.4,6.4) showing that $\alpha = 2$.
- After determining α, consider k. What can in depend on? Perform measurements for various spheres (first a set of measurements for balls of same masses and varying radii, then a set with varying masses and same radii) and determine how k depends on m and r.
  S. (5 points) k can depend on variety of things like air density, shape of the body, mass of the body, external temperature etc. Within the simulation, we need to check how k depends on m and r, or rather how the force depends on those parameters. First we check the dependence on the mass. For this, we need to assure that we compare the forces on the bodies when they have the same velocities. Performing the measurements, it turns out that k does not depend on mass. As for the radius, the author selected $m = 10kg$ and spheres of radii 0.5m, 0.8m and 1m and measured the drag force when the bodies have the same velocities. The measurements give the pairs (radius, force in "meters") = (0.5,0.64), (0.8,1.7), (1,2.76). Applying the same procedure as for the velocity and force we can conclude that k depends on radius quadratically. (a more precise measurement with an exponent fit would also be accepted here).
- (2 points) Select "drag" and then select bodies of same masses and radii, but varying "drag coefficients". Determine how k depends on the drag coefficient.
  S. We can apply the same procedure as in the last assignment. This shows that k depends on the drag coefficient linearly.

Perform a home experiment. Find a ball (a beach ball would be best, but any smooth ball will do), drop it from height h and measure the time required for it to hit the ground. Assuming you know the mass and the radius of the ball, determine k.

S. (10 points) No official solution can be offered here. The students have to measure the height, the time, mass and the radius of the ball and determine k. The determination of k can be done with the help of the simulation - select “drag” and enter the parameters of your ball. Sliding the drag coefficient slider one can alter the fall time in such a way that it coincides with the measured fall time. One should take care of correctly quantifying errors in the experiments.

COUPLE OF IDEAS ABOUT ONE DIMENSIONAL GASES

One dimensional gas

i) [2] By writing law of conservation of energy and momentum, we can see that balls only exchange velocities, i.e. ball which had velocity v_1 before collision will have velocity v_2 after collision and vice versa.

ii) [1] No. Balls change their direction of motion in each collision (or stay stationary) - but basically balls can't pass one through another.

iii) [2] We expect that one ball moves with speed v while others are at rest (velocity can be in the opposite direction because direction of velocity changes on collision with the wall). This is because magnitudes of velocities don't change in collisions. Similarly, in second case, we expect that four balls move with speed $\frac{v}{2}$, while others are at rest. Final case will depend on initial speeds of balls.

Give 1 point if assumed velocity will be distributed equally, and that all particles will have the same energy.

iv) [3] In the frame of the right wall, ball is approaching the wall at velocity $v - u$. Thus after collision in this frame, ball will have same speed but will move in the opposite direction. Returning to original frame we can see that the final velocity of the ball is $v - 2u$. Thus change of energy is simply:

$$\triangle E = \frac{m(v-2u)^2}{2} - \frac{m(v)^2}{2} = -2muv$$

Give 0.5 if energy conservation is mentioned or 1 point if equivalent energy-momentum relationships were written, but equations were solved incorrectly.

v) [2] Mean distance is just equal to the mean distance between two particles, and because there is no difference between particles, we expect something like $\approx \frac{L}{N}$, where L is distance between the walls. Then the time between two collisions is approximately $\approx \frac{2L}{Nv}$, where v is average velocity of a particle. Factor 2 is here because ball hits the wall only in half of the cases (so if its velocity is pointing towards the wall). We can think of it as follows: ball initially has velocity towards the wall so travels distance $\frac{L}{N}$ to hit it. After collision it changes velocity, so it travels another $\frac{L}{N}$ until it changes the direction of its velocity. If the total energy of a gas is E, we expect that it is on average distributed equally between the particles, so $E \approx \frac{Nmv^2}{2}$.

vi) [1] Change of length of the well in time t is $\triangle L = ut$. Thus in a single collision $\triangle L \approx 2\frac{Lu}{Nv}$. Energy changes by $\triangle E = -2muv = -\triangle L\frac{Nmv^2}{L} = -2E\frac{\triangle L}{L}$. $\frac{dE}{E} = -2\frac{dL}{L}$, so $E \propto L^{-2}$.

Collision with the moving wall in the frame of the wall

Slika 1: *

Tools from statistical physics

i) [2] Probability that electron is in state with energy E is $P(E) = \frac{1}{Z}\exp\left(-\frac{E}{k_bT}\right)$, and probability that is in state with energy $-E$ is $P(-E) = \frac{1}{Z}\exp\left(\frac{E}{k_bT}\right)$, where Z is some constant. As total probability must sum to 1 (i.e. electron is either in state with energy E or in state with energy $-E$), then $P(E) + P(-E) = 1$, so $Z = \exp\left(\frac{E}{k_bT}\right) + \exp\left(-\frac{E}{k_bT}\right)$. Average electron energy is just:

$$P(E)\cdot E + (-E)\cdot P(-E) = \frac{E}{Z}\left(\exp\left(-\frac{E}{k_bT}\right) - \exp\left(\frac{E}{k_bT}\right)\right)$$

Give 1 point if formula for average energy correct (even if probabilities are wrong).

ii) [1] Bosons can share state, so we can achieve this in 3 ways - both bosons in state 1, both bosons in state 2, one boson in state 1 and other in state 2. Fermions can't share states, so we can only have one combination - one fermion in state 1 and other in state 2. At $0K$, system occupies lowest energy state - this means that bosons will both be in lower energy states, and fermions will be in the only state they can be - one fermion in state 1 and other in the state 2.

Give 0.5 if solved as if particles were non identical.

iii) [2] For fermions - each fermion chooses one state. We can label fermions with numbers $1,2,3,4,...,N$. Hence first fermion can choose state in m ways, second in $m-1$, etc. Thus we get total of $m(m-1)(m-2)...(m-N)$ states. However, all fermions are identical so we have included too many combinations here. For example, state when fermion 1 is in state 1 and fermion 2 is in state 2 is same to the case when fermion 1 is in state 2 and fermion 2 is in state 1. We can see that we have counted each state $N(N-1)...1$ times, as this is equal to the number of ways in which we can label fermions with numbers 1,2,...N (number 1 chooses fermion in N ways, number 2 in $N-1$ ways etc.). Thus total number of fermion states is

$$\frac{m(m-1)(m-2)...(m-N)}{N(N-1)...1} = \binom{m}{N}$$

As for bosons, they can share states. Thus we can represent states as sticks and bosons as balls. We can put balls between the sticks as we like - if there is 2 balls between sticks denoted by 4 and 5, this means that there is 2 bosons in state denoted by 5. Hence, number of bosons in single state is determined by number of balls left to the corresponding stick (between two sticks). Hence we cannot put any balls to the right of rightmost stick. So in how many ways can we arrange this? Let us consider problem a bit differently. Let us arrange $N+m$ balls on a line. Now we want to turn m balls into sticks. We also know that rightmost element must be a stick, so let's turn it into stick. Now we have to choose $m-1$ out of $N+m-1$ balls to turn into sticks. This is equivalent to problem in first part of question (as order of sticks is irrelevant), so total number of boson states is:

$$\frac{(N+m-1)(N+m-2)(N+m-3)...(N-1)}{(m-1)(m-2)...1} = \binom{N+m-1}{m-1}$$

We expect more different states for bosons as they can share states.

Give 1.5 if solved as if particles were non identical, give all points if just quoted the correct result with reference.

iv) [3] Probability that state with energy E is occupied when there is $N+1$ fermions is

$$P(N+1) = \sum_R \frac{1}{Z}\exp\left(-\frac{E_{tot}}{k_b T}\right)$$

where E_{tot} is total corresponding energy of the system, and Z is just some constant. R means that we sum over cases where there is a fermion in state with energy E. Now by removing fermion from states with energy E, we obtain all N fermion system combinations without filled state with energy E. Thus, energies of a system are now $E_{tot} - E$. Probability that the state with energy E is filled is

$$P(N) = 1 - \sum_R \frac{1}{Z'}\exp\left(-\frac{E_{tot}-E}{k_b T}\right)$$

where Z' is a new normalisation. As N is quite large, removing one fermion doesn't change anything so $Z \approx Z'$, and $P(N) \approx P(N+1)$. Now $P(N) = 1 - \exp\left(\frac{E}{k_b T}\right)\sum_R \frac{1}{Z}\exp\left(-\frac{E_{tot}}{k_b T}\right) \approx 1 - \exp\left(\frac{E}{k_b T}\right)P(N)$. Hence

$$P(N) = \frac{1}{1+\exp\left(\frac{E}{k_b T}\right)}$$

Give 2 points if derivation was quoted, but it was too mathematically complex.

Mass of stars

i) [0.5] Fermions at $0K$ fill states with the lowest possible energy. In each state there can be only one fermion.

ii) [1.5] Fermions will firstly fill states with momentum equal to $\pm\frac{\hbar}{N^{\frac{2}{3}}L}$, then $\pm 2\frac{n\hbar}{N^{\frac{2}{3}}L}$ etc. until we run out of fermions. We have N fermions total so largest value of momentum will be $p_f = \frac{N\hbar}{2LN^{\frac{2}{3}}}$. Corresponding energy is just $E_f = \frac{p_f^2}{2m}$.

Give 1 point if N was used instead of $\frac{N}{2}$, and do not take away any points for errors in further parts which are cause by this error.

iii) [2] Total energy is just the sum of all of the electron energies, i.e. $E_{tot} = 2\cdot(\frac{\hbar}{N^{\frac{2}{3}}L})^2\frac{1}{2m} + 2\cdot 4(\frac{\hbar}{N^{\frac{2}{3}}L})^2\frac{1}{2m} + 2\cdot 9\cdot(\frac{\hbar}{N^{\frac{2}{3}}L})^2\frac{1}{2m} + ... + 2\cdot(\frac{N}{2})^2(\frac{\hbar}{N^{\frac{2}{3}}L})^2\frac{1}{2m} = 2\cdot(\frac{\hbar}{N^{\frac{2}{3}}L})^2\frac{1}{2m}\cdot\frac{\frac{N}{2}(\frac{N}{2}+1)(N+1)}{6} \approx \frac{1}{24m}(\frac{\hbar}{N^{\frac{2}{3}}L})^2N^3 = \frac{1}{24m}(\frac{\hbar}{L})^2N^{\frac{5}{3}}$

iv) [1] To extend a star by a small δL, we have to apply force $-F\delta L = \frac{1}{24m}(\frac{\hbar}{L+\delta L})^2 N^{\frac{5}{3}} - \frac{1}{24m}(\frac{\hbar}{L})^2 N^{\frac{5}{3}} \approx \frac{1}{24m}(\frac{\hbar}{L})^2 N^{\frac{5}{3}}(1-2\frac{\delta L}{L}) - \frac{1}{24m}(\frac{\hbar}{L})^2 N^{\frac{5}{3}} \approx -\frac{1}{12m}(\frac{\hbar}{L})^2 N^{\frac{5}{3}}\frac{\delta L}{L}$. Hence $F = \frac{1}{12m}\frac{\hbar^2}{L^3}N^{\frac{5}{3}}$.

Equivalently, we know that force scales as energy over distance so $F \approx \frac{E_{tot}}{L}$. (Both ways accepted).

v) [1] Force is inversely proportional to mass, and nuclei have much larger mass than electrons (about 2000 times larger). Hence we can ignore contribution due to nuclei.

[2] vi) Gravitational force is approximately $F_g \approx \frac{GM^2}{L^2}$. Ignoring all the numerical factors we get $\frac{1}{m}\frac{\hbar^2}{L^3}N^{\frac{5}{3}} \approx \frac{GM^2}{L^2}$. Using $M \approx Nm_p$ as there is equal number of protons and electrons, we get $M^{\frac{1}{3}}L \approx \frac{\hbar^2}{Gm}m_p^{\frac{5}{3}}$. This gives good agreement with the values for the Sun.

vii) [2] It is necessary to observe net force on body when its radius is $L+\epsilon$. By Second Newton's law, sum of forces is equal to product of mass and acceleration. If acceleration is in the opposite direction than the displacement, force is restoring and system is trying to go back to equilibrium position. This means that equilibrium is stable. If opposite was true, equilibrium would be unstable. Note that

$$\frac{GM^2}{(L+\epsilon)^2} \approx \frac{GM^2}{L^2}\left(1-2\frac{\epsilon}{L}\right)$$

and

$$\frac{1}{m}\frac{\hbar^2}{(L+\epsilon)^3}N^{\frac{5}{3}} \approx \frac{1}{m}\frac{\hbar^2}{L}N^{\frac{5}{3}}\left(1-3\frac{\epsilon}{L}\right)$$

Now net force is:

$$F_{tot} = -\frac{GM^2}{(L+\epsilon)^2} + \frac{1}{m}\frac{\hbar^2}{(L+\epsilon)^3}N^{\frac{5}{3}} \approx \frac{1}{m}\frac{\hbar^2}{L}N^{\frac{5}{3}}\left(1-3\frac{\epsilon}{L}\right) - \frac{GM^2}{L^2}\left(1-2\frac{\epsilon}{L}\right) = -\frac{GM^2}{L^2}\frac{\epsilon}{L}$$

because in equilibrium $\frac{1}{m}\frac{\hbar^2}{L^3}N^{\frac{5}{3}} \approx \frac{GM^2}{L^2}$. We can see that this is a restoring force, so equilibrium is stable. We used negative sign for gravity - this is because it acts in direction opposite to ϵ, so it is a restoring force.

viii) [1] For 1D gas we have force due to kinetic energy $F \approx \frac{E}{L} \approx \frac{Nk_bT}{L} \approx \frac{Mk_bT}{Lm_p}$. This has to be equal to gravity so we get $\frac{Mk_bT}{Lm_p} \approx \frac{GM^2}{L^2}$, and $\frac{M}{L} \approx \frac{Gm_p}{k_bT}$. Using data for Sun we get pretty bad results. Now

$$F_{tot} = -\frac{GM^2}{(L+\epsilon)^2} + \frac{C}{(L+\epsilon)^3} \approx -\frac{GM^2}{L^2}\left(1-2\frac{\epsilon}{L}\right) + \frac{C}{L^3}\left(1-3\frac{\epsilon}{L}\right) = -\frac{GM^2}{L^2}\frac{\epsilon}{L}$$

Here we used the fact that energy scales as $\frac{1}{L^2}$ and C is some constant. We see that this is a restoring force, so equilibrium is stable.

Give full marks if participants do not realize that E scales as $\frac{1}{L^2}$ and hence they arrive at result that equilibrium is unstable.

GALILEO

1. From the moving observer, the objects seems to fall in a straight line, perpendicular to the ground. From the observer who stands still, the object seems to fall following an inclined line (straight line but not perpendicular if air friction is negligible). The moving observer would observe a similar motion if the guy standing still dropped a ball, as the guy standing still since to the point of view of the moving observer, the observer who stands still appears to be moving.

2. It is difficult to pass the ball because it has an initial speed in the direction of movement, and not only on the direction of the threw. (straight line trajectory if air friction is negligible)

3. Flies fly without any problem if the boat is not accelerating (if the windows and doors are closed and therefore there is no draft).

4. This question is quite open (you can do some research on the equivalence principle as example).

5. You are pushed towards the outside direction (outside of the turn). This is because you naturally tend to go in a straight line. In order to deviate you from this straight line, the car exerts a force on you, via the contact between you and your seat/ the car door. This is the force that you feel.

6. The centrifugal force is directed vertically toward the bottom for someone standing in the inside surface of the cylinder, which mimics the gravity. We want $mr\Omega^2 = mg$ which leads to

$$\Omega = \sqrt{\frac{g}{r}} = 1\text{rad/s} \tag{1}$$

so the ship should roughly spin once every 6 seconds, which is much faster than what happens in the movie (it appears to be closer to one turn/ 40-50 seconds in the movie).

7. We feel attracted towards the Earth, so we don't feel this centrifugal force strictly speaking. However it lower the value of g by a bit since it is in the opposite direction to the gravity force: $F = mg - mr\Omega^2 = m(g - 0.034\text{m/s}^2)$.

8. Here it is important to specify in which frame we are working. In the geocentric frame, there is a centrifugal force due to the orbit of the Earth around the sun. Pay attention to units !! $\Omega_{t/s} = 2\pi/(365 \times 24 \times 3600)$ rad/s. $R_{orbit} = 150 \times 10^9$m and $M_{earth} = 6.0 \times 10^24$ kg. So :

$$F_{centrifugal} = M_{earth} \times R_{orbit} \times \Omega_{t/s}^2 = 3.6 \times 10^{22}\text{N}, \tag{2}$$

and we also have

$$F_{gravity}^{s/t} = \frac{G \times M_{earth} \times M_{sun}}{R_{orbit}^2} = 3.6 \times 10^{22}.\text{N} \tag{3}$$

The values are equal, this is why the Earth stay on its orbit : it does not 'fall' in the sun, and it is not ejected by centrifugal force. (*This is pretty much one of Kepler's law.*)

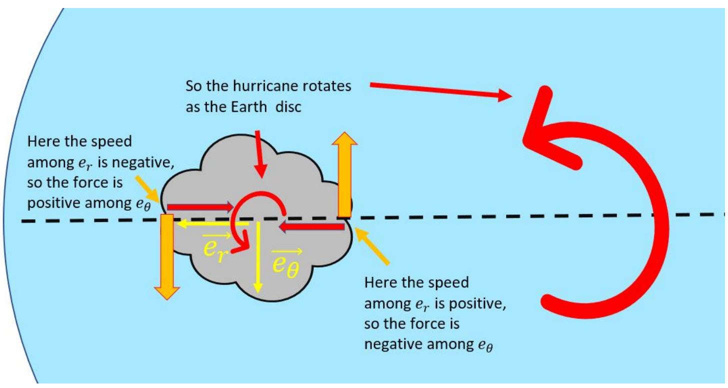
9. To study the motions of the sun, and take into account corrections, we can place ourselves in the galactic reference frame. For the galaxy, this is complex, galaxies are part of a structure called clusters, so we can try to define a cluster class of frame. Another possibility is the reference frame of the cosmic microwave background, but this is quite advanced, and was not expected in the answers. Of course we can easily see with some order of magnitude that the correction due to the sun rotation around the galaxy are extremely small: $r = 2.5 \times 10^{20}$ m, and $\Omega = 8 \times 10^{-16}$ rad/s so $r\Omega^2 = 1.6 \times 10^{-10}$ m/s^2.

10. When the observer is fixed relative to the ground, the ball goes in a straight line. When the observer turns like the merry-go-round, he sees the ball have a curved trajectory. The diagrams are already shown in the video.

11. The rotation of the propeller is the same as the curvature of the balloon's trajectory. If the carousel rotates counterclockwise, the propeller will rotate clockwise.

12. If the earth rotates counterclockwise, for a point close to the center, whose speed is directed towards the center of the cyclone, the coriolis force is directed negatively along the e_θ axis, and it is the opposite for the most distant point. So the cyclone rotates in the same direction as the earth half disk. As depending on the hemisphere, we do not see the earth turning in the same direction, the direction of rotation of a cyclone is reversed for the other hemisphere. An anti-cyclone is like a cyclone except that instead of being attracted to the center of the depression, we are expelled, so the situation of velocity vectors is reversed and in each hemisphere a cyclone turns in the opposite direction of an anticyclone.

13. The answer is no, the coriolis force is negligible for small masses, and we need large objects, oceans, clouds, to observe it. What determines the rotation of water in a sink are: the impurities on the walls, the shape of the sink, the way you wash your hands... (*you can watch on youtube videos that "show" the coriolis effect at the equator, and pretend to observe it with small tanks, and try to find the 'magic trick', sometimes it is obvious* **Here for example**, *sometimes it is more subtle* **Here for example** *!*)



Slika 2: The winds are directed towards the center of the cyclone, which by the formula gives forces that make the whole thing rotate in the same direction as the earth's disk.

Part II

Matematika

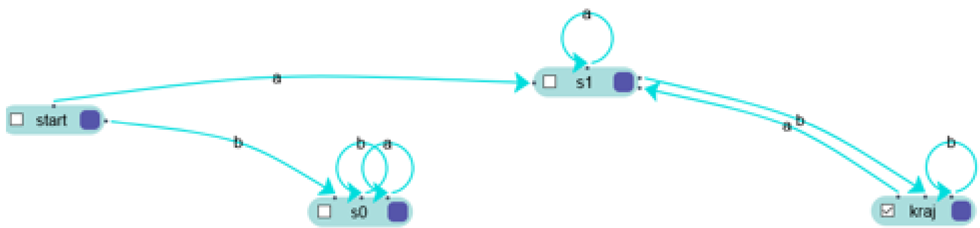
AUTOMATI

(Sipser = Sipser: Introduction to the Theory of Computation, third edition.)

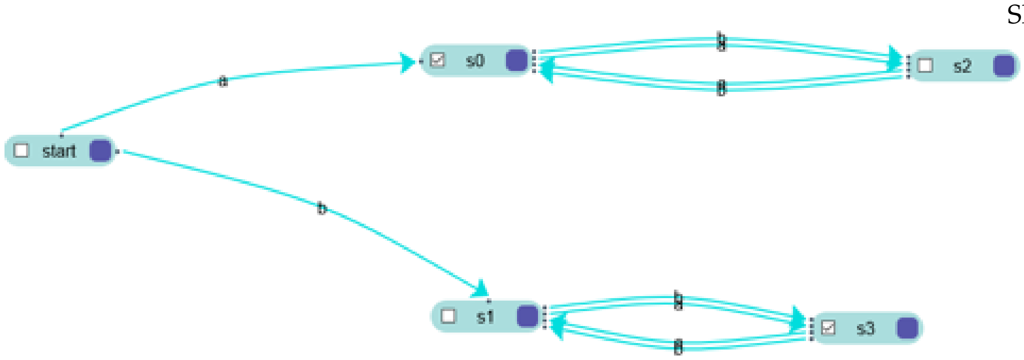
1. Each part 1.5 points.

- Figure 1a: language of all words which have at least three letters `a` (regex: $\Sigma^* a \Sigma^* a \Sigma^* a \Sigma^*$).
- Figure 1b: language of all words which consist solely of either letters `a` or letters `b`, and empty word ε ($a^* \cup b^*$).
- Figure 1c: language of all words which have string `001` as a substring ($\Sigma^* 001 \Sigma^*$).

2. Each part 1.5 points.

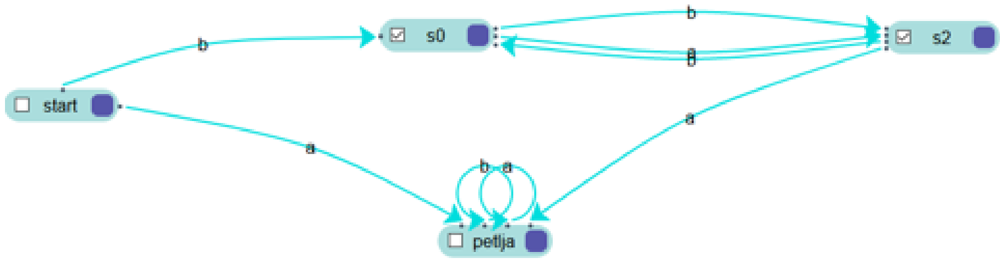


Slika 3: Figure 2a



Slika 4: Figure 2b

3. Sipser, theorem 1.25.



Slika 5: Figure 2c

4. Just swapping the accepting and non-accepting states does the trick.
5. Sipser, theorem 1.47.
6. Sipser, theorem 1.49.
7. Sipser, theorem 1.39.

MATRIČNE AVANTURE

1. 1 point

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + x_2) + b(y_1 + y_2) \\ c(x_1 + x_2) + d(y_1 + y_2) \end{bmatrix} =$$

$$= \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix} + \begin{bmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{bmatrix} =$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

2. 2 + 1 + 1 + 2 points

   (a) $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ is a solution to the given system. For any real number y, $\left[\begin{smallmatrix} 3+2y \\ y \end{smallmatrix}\right]$ is a solution to the given system so the system has infinitely many solutions.

   (b) Let's suppose the system has a solution. Then

$$-6 = 2x - 4y = 2(x - 2y) = 2 \cdot 3 = 6$$

   which gives a contradiction. Hence we conclude that the system has no solutions.

   (c) For any real number y, $\left[\begin{smallmatrix} 2y \\ y \end{smallmatrix}\right]$ is a solution to the given system. Therefore the given system has infinitely many solutions.

   (d) If such a system has a solution $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$ then for any real number y, $\left[\begin{smallmatrix} x_0+2y \\ y_0+y \end{smallmatrix}\right]$ is also a solution to the same system. Hence any such system either has no solutions or it has infinitely many solutions.

3. 1 + 1 point

   (a) $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is clearly a solution to the system. Now let's suppose we have a solution $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ to the system.

$$0 = -p + 3q = -(3x + 5y) + 3(x + 2y) = y$$

$$x = q - 2y = 2p - 5q = 0$$

   (b) If, given p and q, we try to solve for x and y we get:

$$2p - 5q = 2(3x + 5y) - 5(x + 2y) = x$$

$$-p + 3q = -(3x + 5y) + 3(x + 2y) = y$$

A direct computation shows that this really is the solution, hence it is unique. Now we see that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix}$$

hence we've found the desired matrix.

4. 1 point

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}(b_{11}x + b_{12}y) + a_{12}(b_{21}x + b_{22}y) \\ a_{21}(b_{11}x + b_{12}y) + a_{22}(b_{21}x + b_{22}y) \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21})x + (a_{11}b_{12} + a_{12}b_{22})y \\ (a_{21}b_{11} + a_{22}b_{21})x + (a_{21}b_{12} + a_{22}b_{22})y \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

5. 1 point
   Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

   be an arbitrary 2×2 matrix. Then

$$I \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} & 1 \cdot a_{12} + 0 \cdot a_{22} \\ 0 \cdot a_{11} + 1 \cdot a_{21} & 0 \cdot a_{12} + 1 \cdot a_{22} \end{bmatrix} = A$$

$$A \cdot I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 1 \end{bmatrix} = A$$

6. 1 point
   If you have succesfully solved Problem **??** then you should have found that

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

   Direct computation shows:

$$A \cdot B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-5) \cdot 1 & 2 \cdot 5 + (-5) \cdot 2 \\ -1 \cdot 3 + 3 \cdot 1 & -1 \cdot 5 + 3 \cdot 2 \end{bmatrix} = I$$

$$B \cdot A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 5 \cdot (-1) & 3 \cdot (-5) + 5 \cdot 3 \\ 1 \cdot 2 + 2 \cdot (-1) & 1 \cdot (-5) + 2 \cdot 3 \end{bmatrix} = I$$

7. 2 + 1 point

   (a) Suppose A has an inverse

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

   Then from $A \cdot B = I$ we get

$$ax + bz = 1 \qquad ay + bw = 0$$

$$cx + dz = 0 \qquad cy + dw = 1$$

If we try to solve for x we get

$$d = d \cdot 1 - b \cdot 0 = d(ax + bz) - d(cx + dz) = (ad - bc)x$$

If we suppose that $d = 0$, then $c \neq 0$ since $cy = cy + dw = 1$, and hence $cx = cx + dz = 0$ implies $x = 0$. This in turn implies $bz = ax + bz = 1$ so $b \neq 0$. Hence $ad - bc = -bc \neq 0$. On the other hand, if $d \neq 0$, then $0 \neq d = (ad - bc)x$ also gives $ad - bc \neq 0$. This shows that $ad - bc \neq 0$ is a necessary condition for A to have an inverse. Let's show it is sufficient. We've already seen that $d = (ad - bc)x$. If $ad - bc \neq 0$ then

$$x = \frac{d}{ad - bc}$$

Similarly we get

$$y = \frac{-b}{ad - bc}$$
$$z = \frac{-c}{ad - bc}$$
$$w = \frac{a}{ad - bc}$$

A direct computation shows that

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

really is an inverse of A. Our method showed that this matrix is also the only candidate for an inverse of A so we can also conclude that such **inverse is unique**.

(b) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In part (a) of this problem we've shown that A does not have an inverse if and only if $ad - bc \neq 0$. Therefore it is sufficient to prove that the system has infinitely many solutions if and only if $ad - bc = 0$.

Assume that $ad - bc = 0$. Notice that $\left[\begin{smallmatrix} d \\ -c \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -b \\ a \end{smallmatrix}\right]$ are solutions to the system. Now all vector columns of the form

$$\begin{bmatrix} dx - by \\ -cx + ay \end{bmatrix}$$

are solutions to the system. This is an infinite set of solution unless $a = b = c = d = 0$, but in that case every 2×1 matrix is a solution, so the system again has infinitely many solutions.

Now assume that $ad - bc \neq 0$, and let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ be a solution to the system, i.e.

$$ax + by = 0$$

$$cx + dy = 0$$

Now we have

$$0 = d \cdot 0 - b \cdot 0 = d(ax + by) - b(cx + dy) = (ad - bc)x$$

$$0 = -c \cdot 0 + a \cdot 0 = -c(ax + by) + a(cx + dy) = (ad - bc)y$$

Since $ad - bc \neq 0$ we necessarily have $x = y = 0$, hence the solution is necessarily unique, particularly, the system doesn't have infinitely many solutions.

8. 1 + 2 points

(a)

$$\alpha = \frac{x+y}{2}, \beta = \frac{-x+y}{2}$$

(b) One can inductively extend the mentioned identities to

$$A^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$A^n \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)^n \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now, for an arbitrary 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ we have the following.

$$A^n \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^n \cdot \left(\frac{x+y}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-x+y}{2} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = \frac{x+y}{2} \cdot A^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-x+y}{2} \cdot A^n \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

$$= \frac{x+y}{2} \cdot 3^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-x+y}{2} \cdot (-1)^n \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3^n+(-1)^n}{2}x + \frac{3^n-(-1)^n}{2}y \\ \frac{3^n-(-1)^n}{2}x + \frac{3^n+(-1)^n}{2}y \end{bmatrix} =$$

$$\begin{bmatrix} \frac{3^n+(-1)^n}{2} & \frac{3^n-(-1)^n}{2} \\ \frac{3^n-(-1)^n}{2} & \frac{3^n+(-1)^n}{2} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore we have

$$A^n = \begin{bmatrix} \frac{3^n+(-1)^n}{2} & \frac{3^n-(-1)^n}{2} \\ \frac{3^n-(-1)^n}{2} & \frac{3^n+(-1)^n}{2} \end{bmatrix}$$

9. 1 + 1 + 2 + 1 point

(a) A sequence $(a_n)_{n\in\mathbb{N}}$ is geometric if there is a $q \in \mathbb{R}$ such that $a_{n+1} = q \cdot a_n, \forall n \in \mathbb{N}$. If $a_1 = 1$ then $a_n = q^{n-1}$. Let $(a_n)_{n\in\mathbb{N}}$ be a geometric sequence in X and $q \in \mathbb{R}$ associated to it. Now we have

$$0 = a_3 - 2a_2 - 3a_1 = q^2 - 2q - 3 = (q-3)(q+1)$$

Therefore $q \in \{3, -1\}$. Direct computation shows that $(3^{n-1})_{n\in\mathbb{N}}$ and $((-1)^{n-1})_{n\in\mathbb{N}}$ really are in X, and they are indeed the only two sequences sastisfying the wanted conditions.

(b) Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence in X. Then

$$S(a)_{n+2} - 2S(a)_{n+1} - 3S(a)_n = a_{n+3} - 2a_{n+2} - 3a_{n+1} = 0, \forall n \in \mathbb{N}$$

so $S(a)$ is also in X.

(c)

$$\begin{bmatrix} S(a)_1 \\ S(a)_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_2 \\ 3a_1 + 2a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

so

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

(d) Notice that $x_{n+1} = S^n(x)_1$, where $S^n(x) = S(S(...S(x)...))$ (operator S is iterated n times). The matrix A^n corresponds to the operator S^n on X.
Now we want to express an arbitrary $(b_n)_{n\in\mathbb{N}}$ in X in terms of $(3^{n-1})_{n\in\mathbb{N}}$ and $((-1)^{n-1})_{n\in\mathbb{N}}$. Let $x_n = 3^{n-1}$ and $y_n = (-1)^{n-1}$.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 3+(-1) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} S^n(a)_1 \\ S^n(a)_2 \end{bmatrix} = A^n \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = A^n \cdot \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

Notice that

$$A \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and

$$A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = 3^n \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1)^n \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now we see that $a_n = 3^{n-1} + (-1)^{n-1}, \forall n \in \mathbb{N}$.
Remark: Alternatively, one could prove inductively that $a_n = x_n + y_n$ without using matrices, any such proof that is completely valid is worth full marks.

10. 3 + 4 points

(a) Let A^k_{ba} denote the number of paths of length k from vertex a to vertex b. Now we get a simple identity:

$$A^{k+1}_{ba} = \sum_{c=1}^{n} M_{bc} \cdot A^k_{ca}$$

This is because if a path gets from a to c in k steps, then after another step it can end up in b only if $M_{bc} = 1$, and every path that is in b after $k+1$ steps was in some vertex c after k steps. If we extend our definition of multiplication of 2×2 matrices to $n \times n$ matrices by an analogous formula

$$(X \cdot Y)_{ba} = \sum_{c=1}^{n} X_{bc} Y_{ca}$$

and notice that $A^1_{ba} = M_{ba}$ we inductively get that $A^k_{ba} = \left(M^k\right)_{ba}$. Now we are looking for $\left(M^{16}\right)_{3,1}$ in both graphs. The numbers $\left(M^{16}\right)_{3,1}$ for the undirected and directed graphs are 41496 and 129 respectively.

(b) We use a similar idea as in part (a), except that we are looking for **probabilities** that we get from vertex X to vertex Y in k jumps.
Let

$$\begin{bmatrix} P^n_A \\ P^n_B \\ P^n_C \end{bmatrix}$$

represent the probabilities that we are in vertex A, B or C after n jumps. After a single jump, those probabilities change in the following way:

$$\begin{bmatrix} P_A^{n+1} \\ P_B^{n+1} \\ P_C^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{P_B^n + P_C^n}{2} \\ \frac{P_C^n + P_A^n}{2} \\ \frac{P_A^n + P_B^n}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \cdot \begin{bmatrix} P_A^n \\ P_B^n \\ P_C^n \end{bmatrix}$$

because Tony jumps from a vertex to another with probability $\frac{1}{2}$. Let's denote this "transition matrix" with A. Now we use a similar idea as in Problems **??** and **??**. We want to find 3×1 matrices v such that there is a real number λ and $A \cdot v = \lambda \cdot v$.

$$\lambda \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{b+c}{2} \\ \frac{c+a}{2} \\ \frac{a+b}{2} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \lambda \cdot a - \frac{b}{2} - \frac{c}{2} = 0 \\ -\frac{a}{2} + \lambda \cdot b - \frac{c}{2} = 0 \\ -\frac{a}{2} - \frac{b}{2} + \lambda \cdot c = 0 \end{cases}$$

Summing these equations we get:

$$(\lambda - 1) \cdot (a + b + c) = 0$$

If $\lambda = 1$ then necessarily $a = b = c$, and if $a + b + c = 0$ then $\lambda = -\frac{1}{2}$. Now we see that

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -\frac{1}{2} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -\frac{1}{2} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

If Tony starts at vertex A then then $P_A^0 = 1$ and $P_B^0 = P_C^0 = 0$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence

$$\begin{bmatrix} P_A^n \\ P_B^n \\ P_C^n \end{bmatrix} = A^n \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A^n \cdot \left(\frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) =$$

$$= \frac{1}{3} \cdot 1^n \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore

$$P_A^n = \frac{1 + 2 \cdot \left(-\frac{1}{2}\right)^n}{3}$$

is the wanted probability.

SPHERICAL GEOMETRY

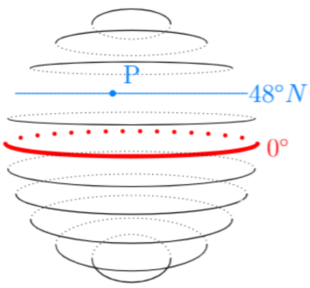
Motivation

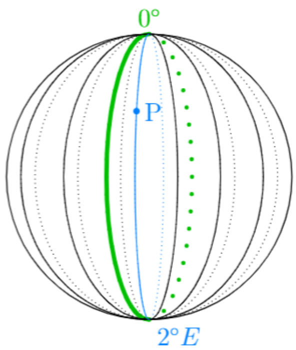
The geometry with which we are most familiar is the Euclidean flat space geometry, whether it be on a 2D plane or in 3D space our intuition serves us well. Changing the shape of the ambient space on which we choose to do geometry can have vast consequences.

In this problem we will taste the flavor of spherical geometry.

Problems of Physical Nature

As with all geometries a natural choice of coordinate axes is needed.





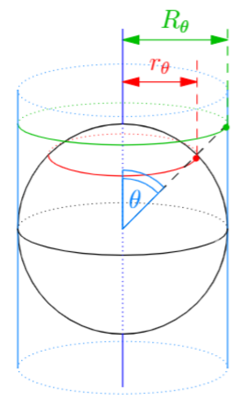
The lattitude and longitude of Paris $P(48,86^\circ N, 2,35^\circ E)$ give us a pair of numbers. The first coordinate says on which parallel north or south of the equator our city lies, while the second tells us on which meridian east or west of the prime meridian (passing through Greenwich) our city lies.

Our first problem arises in the attempt to find an adequate representation of the Earth on a map. No such map is perfect, as distortions are inevitable when representing a 3D object on a 2D plane It is left to you to determine the amount of distance distortion in the two most common maps in cartography.

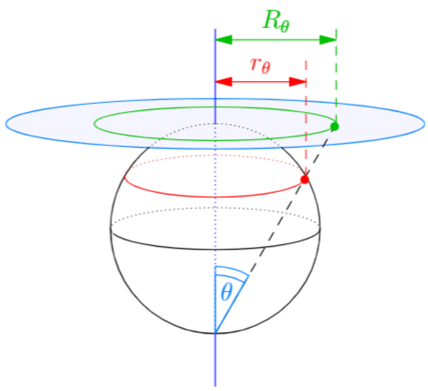
Problem 2.1. (**6 points**) Let Ψ be a mapping from the sphere to another $2D$ surface. Let r_θ be the radius of the parallel at angle $\theta \in [0, \pi/2)$ and R_θ be the radius of its image under Ψ. Determine the ratio $\frac{r_\theta}{R_\theta}$ when

1. (**2 points**)Ψ is the *Mercator* projection, centered at the spheres' center, sending the sphere into a cylinder tangential to the sphere at the equator.

2. (4 **points**)Ψ is the stereographic projection of the sphere, centered at the south pole projecting the sphere onto a plane tangent to the sphere at the north pole.



Cylindrical projection



Stereographic projection

Solution:

1. Let ρ be the radius of our sphere. Because the cylinder is of equal width as we move up along it we can conclude that $R_\theta = \rho, \ \forall \theta \in [0, \pi/2)$. Furthermore we can express r_θ in terms of θ and ρ using the right angled triangle it lives on:

$$sin(\theta) = \frac{r_\theta}{\rho}, \quad \text{and} \quad \rho = R_\theta \qquad \textbf{(1 pt)}$$

Substituting ρ for R_θ we get the following

$$\frac{r_\theta}{R_\theta} = sin(\theta) \qquad \textbf{(1 pt)}$$

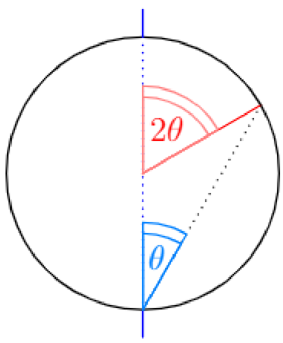
2. Looking into the largest right triangle with angle θ and cathetus R_θ and 2ρ we conclude that

$$tan(\theta) = \frac{R_\theta}{2\rho} \Longrightarrow R_\theta = 2\rho \cdot tan(\theta) \qquad \textbf{(1 pt)}$$

Now it is left to determine r_θ in terms of ρ and θ. In order to do this we must divide the problem into two cases

- if $\theta \in [0, \pi/4)$

, then r_θ can be reached using the radius connecting the center of the sphere and a point on the circle with radius r_θ. Using the fact that the central angle in a circle is twice the circumferential we get that



$$sin(2\theta) = \frac{r_\theta}{\rho} \Longrightarrow r_\theta = \rho \cdot sin(2\theta) \qquad \textbf{(1 pt)}$$

- if $\theta \in [\pi/4, \pi/2)$ then we can use a simple symmetry to shorten our calculations. The parallels at angles θ and $\frac{\pi}{2} - \theta$ have equal radii as one is the mirror image of the other with respect to the apscis. Thus we conclude $r_\theta = r_{\pi/2-\theta} = \rho \cdot sin(\pi - 2\theta) = \rho \cdot sin(2\theta)$ **(1 pt)**

combining the previous results we get

$$\frac{r_\theta}{R_\theta} = \frac{sin(2\theta)}{2tan(\theta)} = \frac{2 \cdot \cancel{sin(\theta)} \cdot cos(\theta)}{2 \cdot \frac{\cancel{sin(\theta)}}{cos(\theta)}} = cos^2(\theta) \qquad \textbf{(1 pt)}$$

Moving from a flat Earth local approximation to a more accurate spherical representation of the Earth has its side effects. The concept of shortest distance must be rethought.

Problem 2.2. (6 points) Zagreb and Toronto are roughly on the same parallel, for the purpose of this problem we approximate their coordinates to be $Z = (45°N, 16°E)$ and $T = (45°N, 79°W)$. Explain why simply heading west along the $45°N$ parallel is not the shortest path from Z to T. How much shorter is the shortest path between these two cities?

Solution

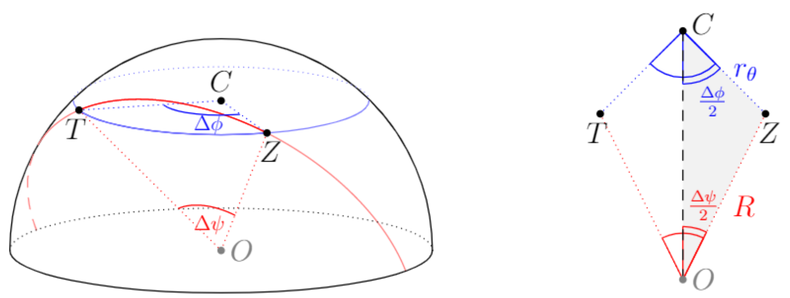
Let $\theta = \theta_Z = \theta_T = 45°N$ and $\phi_Z = 16°E$, $\phi_T = 79°W$. And let $\Delta\phi = 16° + 79° = 95°$ be the meridianal angle between Zagreb and Toronto. If R is the radius of the Earth then the radius of the parallel on which Zagreb and Toronto lie is $r_\theta = R \cdot cos(\theta)$. The length of the arc L_P along this parallel is proportional to $\Delta\phi$ and can be evaluated from the following equality

$$\frac{L_P}{2r_\theta\pi} = \frac{\Delta\phi}{360°} \implies L_P = 2r_\theta\pi\frac{\Delta\phi}{360°} = 2R\pi \cdot cos(\theta)\frac{\Delta\phi}{360} \qquad \textbf{(1 pt)}$$

$$L_P = 2 \cdot 6{,}371km \cdot \pi \cdot cos(45°) \cdot \frac{95°}{360°} \approx 7{,}469km \qquad \textbf{(1 pt)}$$

The shortest path on the sphere is along the arc of a great circle. This fact can be proved using the calculus of variations but we will rely on the intuitive argument that great circles in spherical geometry are analogues of lines in the euclidean plane, as one can conclude in the next section.

Let $\Delta\psi = \angle ZOT$ where O is the center of the Earth. Notice that $\Delta\psi < \Delta\phi$, this is one of the reasons why travelling along the arc of a great circle is shorter.



Unfolding the spatial shape $TCZO$ along the diagonal TZ into a planar quadrangle we get a deltoid. Using the law of sines on triangle CZO we have:

$$\frac{sin(\Delta\psi/2)}{r_\theta} = \frac{sin(\Delta\phi/2)}{R} \implies sin(\Delta\psi/2) = \frac{r_\theta}{R}sin(\Delta\phi/2) = cos(\theta)sin(\Delta\phi/2) \qquad \textbf{(1 pt)}$$

Inserting the numerical values for θ and $\Delta\phi$ we get

$$sin(\Delta\psi/2) = cos(45°)sin(47,5°) \approx 0.5213 \Longrightarrow \Delta\psi \approx 2 * arcsin(0.5213) \approx 62.838° \quad \textbf{(1 pt)}$$

In a similar manner as before , we proceed to find the length L_G of the path along the arc of a great circle:

$$\frac{L_G}{2R\pi} = \frac{\Delta\psi}{360°} \Longrightarrow L_G = 2R\pi\frac{\Delta\psi}{360°} \quad \textbf{(1 pt)}$$

$$L_G \approx 2 \cdot 6,371km \cdot \pi\frac{62.838°}{360°} \approx 6,987km \quad \textbf{(1 pt)}$$

The shortest path is then shorter by

$$L_P - L_G \approx 7,469km - 6,987km \approx 481km \quad \square$$

This shortest distance between two points on a sphere is of key importance, as you shall see in the next section.

Axioms

A great circle is a circle that lies on the surface of a sphere, having the same radius and center as the sphere it lives on. All meridians are for instance great circles, while the only parallel which is a great circle aswell is the equator. Let us now define spherical geometry

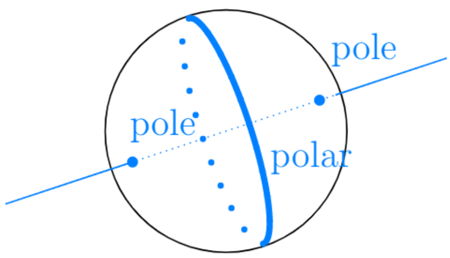
rigorously, with a set of axioms:

1. Any two great circles intersect in two diametrically opposite points
2. Any two points that are not antipodal determine a unique great circle.
3. There is a natural unit of angle measurment (based on revolution), a natural unit of length (based on the circumference of a great circle, $2r\pi$) and a natural unit of area (based on the area of the sphere, $4r^2\pi$)
4. Each great circle is associated with a pair of antipodal points, called its poles which are the common intersections of the great circles perpendicular to it.

Think about it... What could be the analog of great circles in the Euclidean setup.

Polarity

The fourth axiom defines a new relationship between points and great circles (lines in spherical geometry), namely *polarity*. Just as each great circle is associated with a pair of antipodal points. The polar of a point A (and its antipode) is a great circle lying on the bisector plane of A and its antipode.



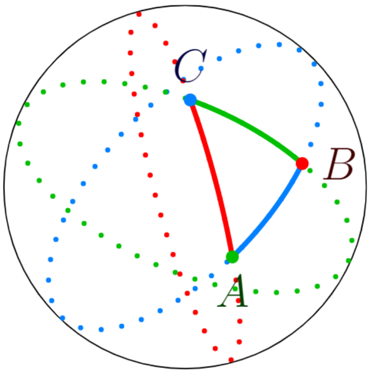
Problem 3.1. (2 points) Prove or disprove: *"If A is on the polar of B, then B is on the polar of A"*

Solution (**La Hire**) Denote the polars of A and B as p_A and p_B respectively. All great circles passing through A are by axiom 4. perpendicular to p_A- Since p_B is one such great circle we conclude $p_A \perp p_B$. Again by axiom 4. all great circles perpendicular to p_B pass through its pole B, as does p_A. Hence p_A passes through B implying that B lies on the polar of A. The converse is analogously true.

Triangles

As in Euclidean geometry, one of the first shapes we encounter is a triangle formed by three lines meeting at three points. We arrive at the concept of a spherical triangle formed by three arcs of distinct great circles, each two arcs intersecting at a vertex of the triangle.

You may use this usefull theorem on the area of spherical triangles.



Theorem The area of a spherical triangle ABC on a unit sphere:

$$S_{ABC} = \angle A + \angle B + \angle C - \pi$$

Where $\angle X$ is the angle at vertex X expressed in radians.

In light of the third axiom, a natural way to think of the angle between two great circles $\mathcal{C}_1$ and $\mathcal{C}_2$ on a sphere is the angle needed to rotate C_1 into C_2

Problem 3.2 (4 points) Prove or disprove : "*$\triangle ABC$ is congruent to $\triangle DEF$ if and only if*

$$\angle A = \angle D \ , \quad \angle B = \angle E \ , \quad \angle C = \angle F \ \text{"}$$

Solution The congruency of two triangles clearly implies that they share the same angles in pairs. The converse also holds, as we shall prove.

Given a spherical triangle $\triangle ABC$, the polar triangle $\triangle A'B'C'$ is the triangle with A a pole of $B'C'$ on the same side as A', with analogous cyclical definitions for B' and C'.

Denote the intersections of AB and AC with $B'C'$ to be X and Y. Because X is on the polar of C' and Y is on the polar of B' we have

$$B'Y = C'X = \frac{\pi}{2} \implies B'C' + XY = B'Y + YC' + XY = B'E + C'D = \pi \qquad \textbf{(1 pt)}$$

As X and Y lie on the polar of A the angular distance XY is equal to $\angle A$, hence :

$$B'C' + \angle A = \pi \qquad \textbf{(1 pt)}$$

Now let $\triangle D'E'F'$ be the polar triangle od $\triangle DEF$, using $\angle A = \angle D$ we have

$$B'C' = \pi - \angle A = \pi - \angle D = E'F' \qquad \textbf{(1 pt)}$$

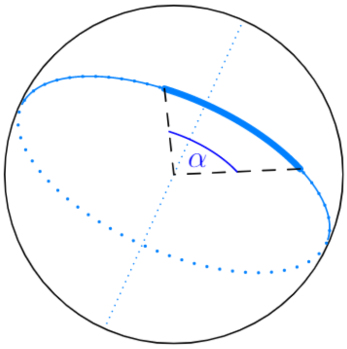
analogously $C'A' = F'D'$ and $A'B' = D'E'$, thus $\triangle A'B'C'$ is congruent to $\triangle D'E'F'$ and these triangles share equal angles in pairs. Using problem 3.1. we find that $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$, the same goes for $\triangle DEF$ being the polar triangle of $\triangle D'E'F'$. We can then apply the previous reasoning with the roles reversed to get

$$BC = \pi - \angle A' = \pi - \angle D' = EF \qquad \textbf{(1 pt)}$$

and analogously $CA = FD$ and $AB = DE$, which means the original triangles truly are congruent. □

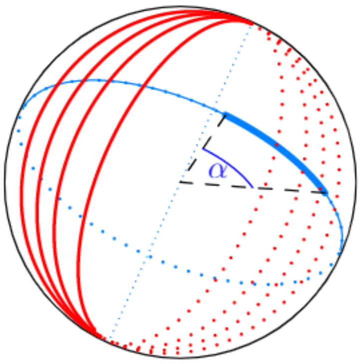
Purely Mathematical results

Problem 4.1. (4 points) Given a spherical line segment of length α on the unit sphere



1. prove that the polars of all spherical lines intersecting this segment sweep out a domain of area 4α
2. find the area swept out by the polars of all the points lying on this segment

Solution Let A be an arbitrary point on this line segment. If A lies on the polar of some point P then by problem 3.1. point P must lie on the polar of A. Hence the poles of all great circles passing through A sweep out the polar of A. **(1 pt)**



The polars of all points lying on this segment share two antipodal points, namely the poles of the great circle our segment lies on. This follows from problem 3.1. aswell. Thus when sweeping point A from one end of our segment to the other its polar sweeps out a spherical lune as shown above. Since the total angular distance the point A has travelled along the given arc is α, the angular distance its polar has travelled must also be α. **(1 pt)**

Notice that parts 1. and 2. are duals of eachother, meaning that in the statement each pole is replaced with its polar and vice versa. The duality of poles and polars can be a very helpful tool while solving harder problems. In this problem we see that the areas in question are in fact the same and finding one leads to the other. **(1 pt)**

Let S_α denote the area of a spherical lune with angle α. Because the area of the unit sphere is 4π, and S_α is proportional to α we have

$$\frac{S_\alpha}{4\pi} = \frac{\alpha}{2\pi} \quad \textbf{(1 pt)}$$

Thus $S_\alpha = 2\alpha$, along with its reflection about the center we have that the total area swept out by the polars of all the points on this segment is 4α □

Problem 4.2. (3 points) Several spherical line segments live on the surface of a unit sphere. If the sum of their lengths is less than π prove the existence of a great circle disjoint from each of the segments.

Solution Denote the lengths of the spherical segments $s_1, s_2, \ldots, s_n$ as $\alpha_1, \alpha_2, \ldots, \alpha_n$ where n is the number of spherical line segments on the unit sphere. By problem 4.1. the poles of all spherical lines interecting segment s_i sweep out a domain of area $4\alpha_i$. **(1 pt)**

Adding these areas yields the area of all the poles whose polars intersects at least one of the segments

$$4\alpha_1 + 4\alpha_2 + \ldots + 4\alpha_n = 4 \cdot (\alpha_1 + \alpha_2 + \ldots \alpha_n) < 4\pi \qquad \textbf{(1 pt)}$$

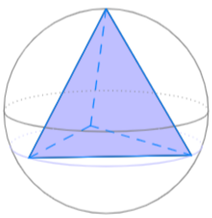
Since this area is less than the total surface area of the unit sphere 4π, there exists a pole whose polar (a great circle) does not intersect any of the given spherical line segments. **(1pt)**

Problem 4.3 (5 points) A triangle T is said to tile the sphere if there is a collection of triangles (spherical) congruent to T that either share a side or vertex or no point at all. Each point of the sphere belongs to a triangle in this collection. Does there exist a triangle T that can tile the sphere with n copies of itself, if:

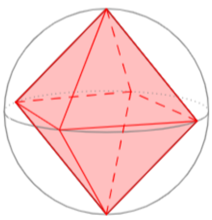
$$a)\ \ n = 4 \qquad\qquad b)\ \ n = 8 \qquad\qquad c)\ \ n = 12$$

If such a T exists, determine its angles.

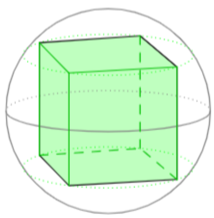
Solution A simple way to generate arbitrary tilings of the sphere (not necessarily triangular) is using the platonic solids. Namely projecting the sides of a inscribed platonic solid onto the sphere through the origin, each side of the platonic solid is projected onto a (spherical) regular polygon that tiles the sphere. Consider the following illustrations



Tetrahedron



Octahedron



Cube

a) Projecting the tetrahedron through the origin onto the sphere generates a tiling with 4 equilateral triangles each with angles 120° (the edges at one vertex divide 360° into three parts). **(1.5 pts)**

b) Projecting the octahedron generates a tiling of 8 equilateral triangles each with angles 90° (the edges at one vertex divide 360° into four parts). **(1.5 pts)**

c) Projecting the cube gives us a tiling with 6 spherical squares with angles 120°, we can use this square tiling to generate a triangular tiling by bisecting each square through one of its diagonals. This generates a tiling of 12 isosceles triangles each with one angle of

120° and two angles 60°. □ **(2 pts)**

In a similar manner the reader can continue to find two more triangular tilings using the remaining two platonic solids, namely the icosahedron and the dodecahedron. The icosahedron naturally generates a tiling with 20 equilateral triangles, while the dodecahedron generates a tiling with 12 equilateral pentagons, further dividing each pentagon as in the case of a cube we get a tiling with 70 triangles.